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Problem 1.109. Let a,b,c be positive real numbers. Prove the inequality

$$\sum \frac{a^3}{b^2+c^2} \ge \frac{a+b+c}{2}.$$

Problem 1.110. Let a,b,c be positive real numbers. Prove that

$$a+b+c \le \sum \frac{a^2+b^2}{2c} \le \sum \frac{a^3}{bc}.$$

Problem 1.111. Let a,b,c be positive real numbers. Prove the inequality

$$\sum \frac{a^2b(b-c)}{a+b} \ge 0.$$

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Problem 1.109.

Solution 1.

By Cauchy Inequality $\sum \frac{a^3}{b^2+c^2} = \sum \frac{a^4}{a(b^2+c^2)} \ge \frac{(a^2+b^2+c^2)^2}{\sum a(b^2+c^2)}$.

Thus, remains to prove $\frac{(a^2+b^2+c^2)^2}{\sum a(b^2+c^2)} \ge \frac{a+b+c}{2} \Leftrightarrow$

$2(a^2+b^2+c^2)^2 \ge (a+b+c)\sum ab(a+b)$.

Let $p := ab+bc+ca, q := abc$. Assuming $a+b+c=1$ (due homogeneity of the inequality) we obtain $a^2+b^2+c^2 = 1-2p$, $\sum ab(a+b) = p-3q$ and, therefore, $2(a^2+b^2+c^2)^2 - (a+b+c)\sum ab(a+b) =$
$2(1-2p)^2 - (p-3q) = 8p^2-9p+3q+2$.

Since $3p = 3(ab+bc+ca) \le (a+b+c)^2 = 1$, $9q \ge 4p-1$ (Schure's Inequality $\sum a(a-b)(a-c) \ge 0$ in p,q notation and normalized by $a+b+c=1$) then

$$8p^2-9p+3q+2 \ge 8p^2-9p+3\cdot\frac{4p-1}{9}+2 = \frac{1}{3}(1-3p)(5-8p) \ge 0.$$

Solution 2.

Since triples (a^3,b^3,c^3) and $\left(\frac{1}{b^2+c^2}, \frac{1}{c^2+a^2}, \frac{1}{a^2+b^2}\right)$ agreed in order then by rearrangement inequality $\sum \frac{a^3}{b^2+c^2} \ge \sum \frac{b^3}{b^2+c^2}$,$\sum \frac{a^3}{b^2+c^2} \ge \sum \frac{c^3}{b^2+c^2}$

and, therefore, $2\sum \frac{a^3}{b^2+c^2} \ge \sum \frac{b^3+c^3}{b^2+c^2}$.

Noting that $\frac{b^3+c^3}{b^2+c^2} \ge \frac{b+c}{2} \Leftrightarrow (b-c)^2 \ge 0$ we obtain

$$2\sum \frac{a^3}{b^2+c^2} \ge \sum \frac{b^3+c^3}{b^2+c^2} \ge \sum \frac{b+c}{2} = a+b+c.$$

Solution 3.

Since triples (a,b,c) and $\left(\frac{a^2}{b^2+c^2}, \frac{b^2}{c^2+a^2}, \frac{c^2}{a^2+b^2}\right)$ agreed in order then by Chebishev's Inequality $\sum \frac{a^3}{b^2+c^2} \ge \sum \frac{a^2}{b^2+c^2}\cdot\frac{a+b+c}{3}$.

Also we have* $\sum \frac{a^2}{b^2+c^2} \ge \frac{3}{2}$(Nesbitt's Inequality).

Hence, $\sum \frac{a^3}{b^2+c^2} \ge \frac{3}{2}\cdot\frac{a+b+c}{3} = \frac{a+b+c}{2}$.

* By Cauchy Inequality

$$\sum \frac{a^2}{b^2+c^2} = \sum\left(\frac{a^2}{b^2+c^2}+1\right)-3 = \frac{1}{2}\sum(b^2+c^2)\sum\frac{1}{b^2+c^2}-3 \geq \frac{9}{2}-3 = \frac{3}{2}.$$

Problem 1.110.

Applying inequality $\frac{x^2}{y} \geq 2x-y \Leftrightarrow (x-y)^2 \geq 0$, where $y > 0$ we obtain:

1. $\sum \frac{a^2+b^2}{2c} = \frac{1}{2}\left(\sum \frac{a^2}{c}+\frac{b^2}{c}\right) \geq \frac{1}{2}\sum(2a-c+2b-c) = \sum(a+b-c) = a+b+c.$

2. $\sum \frac{a^3}{bc} = \sum \frac{a}{b}\cdot\frac{a^2}{c} \geq \sum \frac{a}{b}(2a-c) = 2\sum\frac{a^2}{b}-\sum\frac{ac}{b}$ and

   $\sum \frac{a^3}{bc} = \sum \frac{a}{c}\cdot\frac{a^2}{b} \geq \sum \frac{a}{c}(2a-b) = 2\sum\frac{a^2}{c}-\sum\frac{ab}{c}.$

Hence, $2\sum\frac{a^3}{bc} \geq 2\sum\frac{a^2}{b}+2\sum\frac{a^2}{c}-\sum\frac{ac}{b}-\sum\frac{ab}{c} =$

$2\sum\frac{a^2+b^2}{c}-2\sum\frac{ab}{c} \Leftrightarrow \sum\frac{a^3}{bc} \geq \sum\frac{a^2+b^2}{c}-\sum\frac{ab}{c} =$

$\sum\frac{a^2+b^2}{2c}+\sum\left(\frac{a^2+b^2}{2c}-\frac{ab}{c}\right) = \sum\frac{a^2+b^2}{2c}+\sum\frac{(a-b)^2}{2c} \geq \sum\frac{a^2+b^2}{2c}.$

Problem 1.111.

Noting that $\sum\frac{a^2b(b-c)}{a+b} \geq 0 \Leftrightarrow \frac{1}{abc}\sum\frac{a^2b(b-c)}{a+b} \geq 0 \Leftrightarrow \sum\frac{ab-ca}{ca+bc} \geq 0$

and denoting $x := bc, y := ca, z := ab$ we obtain $\sum\frac{ab-ca}{ca+bc} = \sum\frac{z-y}{x+y}.$

Since triples (x,y,z) and $\left(\frac{1}{y+z},\frac{1}{z+x},\frac{1}{x+y}\right)$ agreed in order then by

Rearrangement inequality $\sum\frac{z}{x+y} \geq \sum\frac{y}{x+y} \Leftrightarrow \sum\frac{z-y}{x+y} \geq 0.$

Hence, $\sum\frac{ab-ca}{ca+bc} \geq 0.$